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STORAGE PROBLEMS WHEN DEMAND IS "ALL OR NOTHING"

by

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and
P. A. Jacobs

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# STORAGE PROBLEMS WHEN DEMAND IS "ALL OR NOTHING"

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## 0. ABSTRACT

An inventory of physical goods or storage space (in a communications system buffer, for instance) often experiences "all or nothing" demand: if a demand of random size $D$ can be immediately and entirely filled from stock it is satisfied, but otherwise it vanishes. Probabilistic properties of the resulting inventory level are discussed analytically, both for the single buffer and for multiple buffer problems. Numerical results are presented.

## 1. INTRODUCTION

The usual storage or inventory problems involve demands imagined to occur randomly, and to be capable of reducing any available stock to zero, or even beyond when backordering is permitted. Yet in many situations at least one component of total demand is "all or nothing;" that is, it reduces inventory only if it can be entirely satisfied by the inventory present, and otherwise seeks another supplier. Here are examples.

(a) A manufacturer's warehouse is filled with a certain item at the beginning of the selling season; let $I$ denote the initial inventory. Suppose that demands occur as follows: a message is sent requesting that $D_1$ items be shipped from inventory, but only if the entire order can be filled. That is, the demand is satisfied if $D_1 \leq I$, in which case inventory level is reduced to $I(1) = I - D_1$; while if $D_1 > I$ the inventory remains unchanged and $I(1) = I$. Allowing for no replenishment, the second demand, of size $D_2$, interacts with inventory $I(1)$, so that it is filled if $D_2 \leq I(1)$, but is not placed if $D_2 > I(1)$. The process continues along these lines until the selling season is over and there are no more demands.

(b) A buffer storage device used to contain messages prior to their batch transmission has capacity $I$. Messages of length $\{D_i, i = 1,2,...\}$ approach the buffer successively, and are admitted on an "all or nothing" basis, just as was true of demands for physical inventory in (a) above. Once again rejection will occur, and more frequently to large demands (messages) than to short ones.

(c) A system of many buffer storage devices is used to contain messages prior to their batch transmission. Each buffer has capacity $I$. Messages of length $\{D_i, i = 1,2,...\}$ approach the device and are successively admitted to the

first buffer until there is a demand that exceeds its remaining capacity. The first buffer is left forever and the demand that exceeds the first buffer, plus successive demands, apply to the second buffer until one occurs that exceeds the remaining capacity. This demand then applies to the third buffer, and so on. As a result there will be some unused capacity in each buffer.

In Section 2 we will discuss some models for the situations in examples (a) and (b). We will compute such quantities as the distribution of the amount of inventory left at some time t and the distribution of the times of successive unsatisfied demands.

In Section 3 we will consider a model for example c. We will derive equations for the limiting distribution of used capacity of a buffer and the expected used capacity of a buffer. It seems to be difficult to obtain simple analytic solutions to these equations, but we will present certain illustrative numerical results.

## 2. THE ONE-BUFFER INVENTORY PROBLEM

Suppose that demands for available stock occur according to a compound Poisson process: if $N_t$ is the number of demands that occur in $(0,t]$, then $\{N_t; t \geq 0\}$ is a stationary Poisson process with rate $\lambda$; the sizes of successive demands $\{D_i\}$ are independent with common distribution $F$. Assume that there are no replishments of inventory. Let $\{I_t; t \geq 0\}$ denote the stochastic process describing available inventory at time $t$, and let $\{I(n); n = 0,1,\ldots\}$ be the stochastic process of available inventory following the nth demand. It is apparent from our assumptions that both $\{I_t\}$ and $\{I(n)\}$ are Markov processes.

### 2.1. Functional Equations for the Amount of Available Inventory

Let

$$\phi(s,t) = E[e^{-sI_t}] \tag{2.1}$$

be the Laplace transform of the available inventory at time t. Similarly, let

$$\psi(s,n) = E[e^{-sI(n)}] \ .$$

Properties of the available inventory can be studied in terms of $\phi$ and $\psi$; we begin by deriving an equation for $\phi$.

Observe that if one conditions on $I_t$, then in $(t, t+dt)$ either no change in inventory occurs, an event of probability $1 - \lambda F(I_t)dt + o(dt)$, or a depletion of amount $x$ occurs with probability $\lambda\, dt\, F(dx)$, $x \leq I_t$. Thus

$$\text{(2.2)}\quad E[\exp(-sI_{t+dt})\,|\,I_t]$$
$$= e^{-sI_t}[1 - \lambda F(I_t)dt] + \int_0^{I_t} \exp[-s(I_t - x)]\lambda dt\, F(dx) + o(dt) \quad .$$

Now take expectations with respect to $I_t$:

$$\text{(2.3)}\quad \phi(s,\; t+dt)$$
$$= \phi(s,t) - \lambda\{E[e^{-sI_t}F(I_t)] - E[\int_0^{I_t} \exp\{-s(I_t - x)\}\, F(dx)]\}\, dt + o(dt).$$

After subtraction of $\phi(s,t)$ from both sides, division by $dt$, and allowing $dt \to 0$ we find that $\phi$ must satisfy the equation

$$\text{(2.4)}\qquad \frac{\partial\phi}{\partial t} = \lambda E\left[e^{-sI_t}\int_0^{I_t} (e^{sx} - 1)\; F(dx)\right].$$

An analogous argument shows that

$$\text{(2.5)}\qquad \psi(s,\; n+1) = \psi(s,n) + E[e^{-sI(n)}\int_0^{I(n)} (e^{sx} - 1)\; F(dx)].$$

Differentiation with respect to $s$ at $s = 0$, or a direct conditional probability argument, now produce equations for $E[I_t]$ and $E[I(n)]$:

$$\text{(2.6)} \qquad \frac{d}{dt} E[I_t] = -\lambda E[\int_0^{I_t} x\, F(dx)]$$

and

$$E[I(n+1)] = E[I(n)] - E[\int_0^{I(n)} x\, F(dx)] \; .$$

In general no explicit solutions for the expected values are available, but a simple upper bound results from rewriting (2.6) as follows.

$$\text{(2.7)} \qquad \begin{aligned} \frac{d}{dt} E[I_t] &= -\lambda E[I_t \int_0^{I_t} \frac{x}{I_t} F(dx)] \\ &\geq -\lambda E[I_t\, F(I_t)] \\ &\geq -\lambda F(I)\, E[I_t] \; , \end{aligned}$$

from which one sees that

$$\text{(2.8)} \qquad E[I_t] \geq I \exp[-\lambda F(I) t]$$

and similarly

$$E[I(n)] \geq I[1 - F(I)]^n \; ,$$

so the expected available inventory declines by at most an exponential rate.

## 2.2. Explicit Solution When the Demand Distribution is Uniform

Although Equation (2.4) seems to be quite intractable for most demand distributions, it can be solved completely when $F$ is uniform:

$$F(x) = \begin{cases} \frac{x}{c} & 0 \le x \le c, \\ 1 & c \ge x \end{cases}$$

and $c \ge I$. In this case (2.4) can be expressed as

$$\text{(2.9)} \qquad \frac{\partial\phi}{\partial t} = \lambda E\left[e^{-sI_t} \int_0^{I_t} (e^{sx} - 1)\, \frac{dx}{c}\right]$$

$$= \lambda E\left[\frac{1 - e^{-sI_t}}{sc} - \frac{e^{-sI_t} I_t}{c}\right]$$

$$= \frac{\lambda}{c}\left[\frac{1 - \phi}{s}\right] + \frac{\lambda}{c}\, \frac{\partial\phi}{\partial s}\ .$$

In other words $\phi$ satisfies a first-order (quasi) linear partial differential equation with initial condition $\phi(s,0) = e^{-sI}$. The usual procedure for solution, Sneddon (1957), requires solution of two ordinary differential equations selected from among

$$\text{(2.10)} \qquad \frac{dt}{1} = \frac{-ds}{(\lambda/c)} = \frac{d\phi}{(\lambda/c)\,[(1-\phi)/s]}\ ;$$

we find from the first and last two that

(2.11) $$s + \frac{\lambda}{c} t = c_1 , \qquad \frac{1 - \phi}{s} = c_2 ;$$

so a general solution is given by

(2.12) $$g\left(s + \frac{\lambda}{c} t, \ \frac{1 - \phi}{s}\right) = 0 ,$$

$g$ being a function to be determined. Now the specified initial condition stipulated that at $t = 0$

(2.13) $$\phi - e^{-sI} = 0 = g\left(s, \frac{1 - \phi}{s}\right) ,$$

so

(2.14) $$\frac{1 - \phi}{s} - \frac{1 - e^{-sI}}{s} = 0$$

which specifies $\phi$ at $t = 0$. But for $t$ positive we replace $s$, the first argument of $g$ at $t = 0$ by $s + (\lambda/c)t$ to obtain the solution

(2.15) $$\frac{1 - \phi(s,t)}{s} - \frac{1 - \exp[-(s + (\lambda/c)t)I]}{s + (\lambda/c)t} = 0 ,$$

which gives the desired transform. Passage to the limit as $s \to 0$ in (2.15) shows that

$$(2.16) \qquad E[I_t] = \frac{1 - \exp[-(\lambda t/c)I]}{(\lambda/c)t} .$$

This formula can also be derived by first finding an expression for the k<u>th</u> moment of $I_t$, and then employing a Taylor series argument.

In order to invert the transform in (2.15) note that

$$(2.17) \qquad \int_0^I e^{-sx} P\{I_t > x\}dx = \frac{1 - \phi(s,t)}{s} = \frac{1 - \exp[-(s + (\lambda t/c))I]}{s + (\lambda/c)t}$$

which is the transform of a truncated exponential distribution. Thus by the unicity theorem for Laplace transforms

$$(2.18) \qquad P\{I_t > x\} = \begin{cases} \exp[-(\lambda t/c)x] & 0 \le x < I, \\ 0 & I \le x . \end{cases}$$

Note that the distribution of $I_t$ is absolutely continuous in the interval $(0,I)$ but that there is a jump at $I$ corresponding to the occurrence of no demand less than or equal to $I$ in $(0,t]$:

$$(2.19) \qquad P\{I_t = I\} = \exp[-\lambda t(I/c)] .$$

## 2.3. The Expected Number of Satisfied Demands

Supposing that an initial inventory, or storage capacity, $I$ prevails, it is of interest to compute the probability that a demand is satisfied, and the expected number of demands satisfied in an interval of length $t$. First notice that if a demand of size $D(t)$ appears at time $t$, at which moment $I_t$ is available, then

$$P\{D(t) < I_t | I_t\} = F(I_t)$$

is the conditional probability that the demand is satisfied. When $F$ is uniform, as is presently true, we may remove the condition to find that

$$P\{D(t) \leq I_t\} = E[F(I_t)] = E\left[\frac{I_t}{c}\right] = \frac{1 - \exp[-(\lambda t/c)I]}{\lambda t} .$$

If $S(t)$ is the number of demands satisfied during the time interval $(0,t]$, then since demands arrive according to a Poisson process with rate $\lambda$,

$$(2.20) \quad E[S(t)] = \lambda \int_0^t E[F(I_u)]du = \lambda \int_0^t \frac{1 - \exp[-(\lambda u/c)I]}{\lambda u} du$$

$$= \gamma + \ell n(\frac{\lambda t I}{c}) + E_1(\frac{\lambda t I}{c})$$

where $E_1(\cdot)$ is an exponential integral; Abramowitz and Stegun (1965), and $\gamma = 0.5112...$ is Euler's constant.

## 2.4. The Time of the First Unsatisfied Demand and the Amount of Unused Inventory at That Time

As before $F$ is the common distribution function of the successive demands. Now let $\tau$ be the time of the first unsatisfied demand. Then

$$P\{\tau > t | N_t = n\} = P\{D_1 \leq I, D_2 \leq I - D_1, \ldots, D_n \leq I - D_1 - \cdots - D_{n-1}\}$$

$$= F^{(n)}(I)$$

where $F^{(n)}$ denotes the nth convolution of $F$ with itself. Hence

$$P\{\tau > t\} = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} F^{(n)}(I) . \tag{2.21}$$

Explicit expressions for the distribution of $\tau$ can be obtained in some cases. If $F$ is uniform on $[0,c]$ with $c \geq I$, then

$$P\{\tau > t\} = e^{-\lambda t} I_0 \left[2\left(\frac{\lambda t I}{c}\right)^{1/2}\right] \tag{2.22}$$

where $I_0(z)$ is a modified Bessel function of the first kind of the zeroth order. In this case

$$E[\tau] = \frac{1}{\lambda} \exp\{I/c\} = \frac{1}{\lambda} \exp\{I/2E[D]\} . \tag{2.23}$$

If $F$ is <u>exponential</u> with mean $1/\mu$, then

$$(2.24) \qquad P\{\tau > t\} = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=n}^{\infty} e^{-\lambda I} \frac{(\lambda I)^k}{k!}$$

and

$$(2.25) \qquad E[\tau] = \frac{1}{\lambda} [1 + \mu I] = \frac{1}{\lambda} [1 + \frac{I}{E[D]}] .$$

Note that if $I$ is small relative to $E[D]$, then the expected time to first unsatisfied demand when $F$ is exponential will be greater than the expected time when $F$ is uniform. However for $I$ large relative to $E[D]$ the expected time for $F$ exponential will be less than the expected time when $F$ is uniform.

Let $Y_n$ be the amount of inventory present at the time of the $n^{th}$ unsatisfied demand. Then for $0 \leq a \leq I$

$$(2.26) \qquad P\{Y_1 \geq I - a\} = \int_0^a R(dy)\, \bar{F}(I - y)$$

where

$$(2.27) \qquad R(y) = \sum_{n=0}^{\infty} F^{(n)}(y)$$

and

$$(2.28) \qquad \bar{F}(I - y) = 1 - F(I - y).$$

Again explicit expressions for the distribution of $Y_1$ can be obtained for some distributions $F$. If $F$ is uniform on $[0,c]$ for $c \geq I$, then

$$(2.29) \qquad P\{Y_1 \geq I - a\} = 1 - \left(\frac{I - a}{c}\right) \exp\{\frac{1}{c} a\} .$$

If $F$ is a truncated exponential

$$(2.30) \qquad F(x) = \begin{cases} \dfrac{1 - e^{-\mu x}}{1 - e^{-\mu I}} & x \leq I , \\ \\ 1 & x \geq I , \end{cases}$$

then

$$(2.31) \quad P\{Y_1 \geq I-a\} = 1 - [e^{-\mu a} - e^{-\mu I}][1 - e^{-\mu I}]^{-1} \exp\{\mu a[1-e^{-\mu I}]^{-1}\} .$$

If $F$ has an exponential distribution with mean $1/\mu$, then

$$(2.32) \qquad P\{Y_1 \geq I - a\} = e^{-\mu(I-a)} .$$

In this last case the distribution function of $Y_n$ can be computed by induction quite easily and

$$(2.33) \qquad P\{Y_n \geq I - a\} = e^{-n\mu(I-a)} .$$

Hence when $F$ is exponential

$$(2.34) \qquad E[Y_n] = \frac{1}{n\mu} [1 - e^{-n\mu I}] .$$

In principle similar results can be obtained for other distributions, but we have found no simple expressions.

## 2.5. Inventory Costs and Policies

There are at least three monetary quantities which affect the profitability of an inventory policy over a fixed interval of time $(0,t]$: the selling price, $p$; the storage cost, $a$; and the cost of lost demands, $b$. If the storage cost $a$ is charged just on the basis of $I$ (something like warehouse size) then the total expected profit in $(0,t]$ is

$$
\begin{aligned}
Z(I) &= p(I - E[I_t]) - aI - bS(t) \\
&= (p - a)I - p\left(\frac{\lambda}{c}\, t\right)^{-1} [1 - \exp[-(\lambda t/c)I]] \\
&\quad - b\{\gamma + \ell n(\tfrac{\lambda t}{c}\, I) + E_1(\tfrac{\lambda t}{c}\, I)\}
\end{aligned}
$$

for the case of uniformly distributed demands; see ((2.16) and (2.20)). One can numerically find the maximum expected profit for this case; nothing explicit seems to be available.

## 3. THE MANY-BUFFER STORAGE PROBLEM

In this section we will study a model for the situation of example (c) in section 1. Messages are successively admitted to the nth buffer until there is a message length that exceeds the remaining capacity of the buffer. The total amount of this message is put in the (n+1)st buffer and the nth buffer is left forever. Successive messages are then put in the (n+1)st buffer until there is a message whose length exceeds the remaining capacity of the (n+1)st buffer; this message is put in the (n+2)nd buffer and so on.

Let $I$ denote the common capacity of the buffers and $D_i$ denote the length of message i. Assume $\{D_i\}$ is a sequence of independent identically distributed random variables with distribution $F$ having a density function $f$ such that $f(x) \geq d > 0$ for $x \in [0,I]$. Let $R(x) = \sum_{n=0}^{\infty} F^{(n)}(x)$ be the renewal function associated with $F$. If $F(I) < 1$, then we will assume that an incoming message to the currently used nth buffer of length greater than $I$ is sent to the (n+1)st buffer; when it cannot fit into the (n+1)st buffer, then it is "banished," i.e. sent to some other set of buffers. The next message however will try to enter the (n+1)st buffer. If this message has length greater than $I$ it is banished and the following message will try to enter the (n+1)st buffer; all messages of length exceeding $I$ will be banished until one appears that is smaller than $I$ and it will be the first entry in buffer (n+1).

This model has been studied for demand distributions $F$ with $F(I) = 1$ by Coffman et al. (1978). Their approach was to study the Markov process describing the total amount of inventory or space consumed in successive buffers or bins. Here we study the process $\{L_n\}$, where $L_n$ is the size of the demand that first exceeds the remaining capacity of the <u>nth</u> buffer; $\{L_n;\ n = 1,2,\ldots\}$ is a Markov process. Let

$$K(x, [0,y]) = P\{L_{n+1} \leq y | L_n = x\}.$$

Note that

$$P\{L_1 \leq y\} = K(0, [0,y])$$

is the same as the sum of the forward and backward recurrence times at time $I$ for a temporal renewal process with inter-renewal distribution $F$. Thus for $y \leq I$

$$\text{(3.1)} \qquad H_1(y) \equiv P\{L_1 \leq y\} = \int_{I-y}^{I} R(dz)\ [F(y) - F(I-z)].$$

Note that for $y < I$

$$\text{(3.2)} \quad K(x,[0,y]) = \begin{cases} \displaystyle\int_{I-x-y}^{I-x} R(dz)\,[F(y)-F(I-x-z)] & \text{if } x < I-y; \\[2ex] \displaystyle\int_{0}^{I-x} R(dz)\,[F(y)-F(I-x-z)] & \text{if } I-y \leq x < I; \\[2ex] \displaystyle\int_{I-y}^{I} R(dz)\,[F(y)-F(I-z)] & \text{if } x > I. \end{cases}$$

Hence

$$
(3.3)\quad K(x,dy) = \begin{cases} [R(I-x) - R(I-x-y)]\ F(dy) & \text{if } x < I-y, \\ R(I-x)\ F(dy) & \text{if } I > x > I-y, \\ R(y)\ F(dy) - \int_0^y R(dz)\ f(y-z) + R(dy)F(y) & \\ & \text{if } x = I-y\ , \\ [R(I) - R(I-y)]\ F(dy) & \text{if } x > I\ . \end{cases}
$$

Note that for some $0 < a < b < I$, there exists a $\delta > 0$ such that for all $x$

$$K^2(x,dy) \geq \delta \qquad \text{for} \qquad y \in [a,b]$$

where $K^2(x,dy) = \int_0^\infty K(x,dz)\ K(z,dy)$. Hence hypothesis D' on page 197 of Doob (1952) is satisfied. Thus, if

$$K^n(x,A) = P\{L_{1+n} \in A | L_1 = x\}$$

for all Borel subsets $A$, then

$$(3.4)\qquad \lim_{n \to \infty} K^n(x,A) = H(A)$$

exists and further the convergence is geometric

$$|K^n(x,A) - H(A)| \leq \alpha\gamma^n$$

for some positive constants $\alpha$ and $\gamma$, $\gamma < 1$ for all $A$.

Now let

$$H_n(x) = P\{L_n \in [0,x] | L_0 = 0\}.$$

Then a renewal argument can be used to show that for $x \leq I$

$$(3.5) \qquad H_{n+1}(x) = \int_{I-x}^{I} H_n * R(dy)\ [F(x) - F(I-y)]$$

$$+ [1-H_n(I)] \int_{I-x}^{I} R(dy)\ [F(x) - F(I-y)] .$$

Taking limits as $n \to \infty$ it is seen that the distribution $H(x)$ satisfies the following equation for $x \leq I$:

$$(3.6) \qquad H(x) = \int_{I-x}^{I} H * R(dy)\ [F(x) - F(I-y)]$$

$$+ [1-H(I)] \int_{I-x}^{I} R(dy) [F(x) - F(I-y)] .$$

Equations (3.1) and (3.6) can be simplified for certain specific distributions $F$.

A. Exponential Demands.

For the exponential distribution with mean 1 and $x \leq I$ the equations are

$$(3.7) \qquad H_1(x) = 1 - e^{-x} - xe^{-x}$$

and

$$(3.8) \qquad H(x) = xe^{-x} H(I) + H_1(x) - e^{-x} \int_0^x H(I-x+u)du .$$

B. Uniform Demands

For the uniform distribution on $[0,c]$ with $c \geq I$ they simplify to

$$(3.9) \qquad H_1(x) = \exp[\tfrac{1}{c}(I-x)] - (1 - \tfrac{x}{c}) \exp(\tfrac{1}{c} I)$$

and

$$(3.10) \qquad H(x) = \tfrac{1}{c} \exp[\tfrac{1}{c}(I-x)] \int_0^{I-x} \exp(-\tfrac{1}{c} u)\, H(u)\,du$$
$$+ \tfrac{x}{c} H(I) - \tfrac{1}{c} \exp(\tfrac{1}{c} I)\,(1 - \tfrac{x}{c}) \int_0^{I} \exp(-\tfrac{1}{c} u)\, H(u)\,du$$
$$+ [1 - H(I)]\, H_1(x) ,$$

for $x \leq I$. Similar expressions hold for $x > I$, but they are unimportant in the present context.

Equations (3.6), (3.8) and (3.10) do not seem to yield explicit answers. As a result we have solved (3.8) and (3.10) numerically by iteration using the system of equations

$$(3.11) \qquad H_{n+1}(x) = xe^{-x} H_n(I) + H_1(x) - e^{-x} \int_0^{x} H_n(I-x+u)\,du$$

with $H_1$ as in (3.7) and

$$(3.12) \qquad H_{n+1}(x) = \tfrac{1}{c} \exp[\tfrac{1}{c}(I-x)] \int_0^{I-x} \exp(-\tfrac{1}{c} u)\, H_n(u)\,du$$
$$+ \tfrac{x}{c} H_n(I) - \tfrac{1}{c} \exp(\tfrac{1}{c} I)(1 - \tfrac{x}{c}) \int_0^{I} \exp(-\tfrac{1}{c} u)\, H_n(u)\,du$$
$$+ [1 - H_n(I)]\, H_1(x)$$

with $H_1$ as in (3.9). For the cases carried out the convergence

is rapid; after $n = 5$ iterations very little change is noted and convergence has occurred, for most practical purposes.

Next let $Y_n$ be the amount of storage space used in the n<u>th</u> bin; the distribution of $Y_n$ is denoted by $G_n(x)$, and

$$G(x) = \lim_{n \to \infty} P\{Y_n \leq x\} = \lim_{n \to \infty} G_n(x)$$

is the long-run distribution. By probabilistic arguments and (3.4)

$$(3.13) \quad G(x) = \int_0^x H * R(dy)\, \bar{F}(I-y) + [1-H(I)] \int_0^x R(dy)\, \bar{F}(I-y)$$

where $\bar{F}(I-y) = 1 - F(I-y)$ and the long run average expected capacity of a bin that is actually used is

$$A = \int_0^I x\, G(dx) .$$

For the case in which $F$ is exponential with unit mean

$$(3.14) \qquad A = I - [1-H(I)][1-e^{-I}] - e^{-I} \int_0^I e^x H(x)dx .$$

For the case in which $F$ is uniform on $[0,c]$ with $c \geq I$

$$(3.15) \quad A = -2 \int_0^I H(u)du + \exp(\tfrac{1}{c} I) \int_0^I \exp(-\tfrac{1}{c} u)\, H(u)du$$

$$+ H(I)\, [2I - c \exp(\tfrac{1}{c} I) + c] + [-I + c \exp(\tfrac{1}{c} I) - c].$$

Numerical solutions were obtained for equations (3.14) and (3.15) by first computing the probabilities $H_n(x)$, $n = 1,2,...,10$ iteratively from (3.7) and (3.11) for the exponential demand case, and from (3.9) and (3.12) for the case of uniform demands. Our technique was simply to discretize x: $x_j = jh$, $h = I/N$, N being the number of x-values at which $H_n(x)$ is evaluated (values of N from 200-1200 were utilized in order to obtain two-significant digit accuracy). The integrals were then approximated by a summation, i.e. Simpson's rule. Having the values of $H_n(x_j)$ it is possible to calculate those of $H_{n+1}(x_j)$, and from these the values of $G_n(x)$ and the mean usage, $E[Y_n]$, may be calculated by numerical integration. In the case of exponential demand very simple upper and lower bounds were obtainable; such bounds were not tight enough to be useful for the uniform case.

The following table summarizes the numerical results. We have compared demand distributions that result, as nearly as possible, in the same probability that an initial demand on an empty bin will be rejected. We have tabulated the expected level to which the bin is filled. It is interesting that the limited bin occupancy is 0.75 when a uniform demand over the range of the bin size is experienced. This result has been obtained analytically by Coffman et al. (1978); in that paper simple and elegant analytical expressions for G and H also appear for this case. The considerable similarity of the numbers in the rows of the table is notable; apparently the long-run bin occupancy is only slightly larger than is that

of the first bin, and the occupancy experienced for uniform demand is only slightly larger than for exponential. Further investigations to examine the reasons for this insensitivity would seem to be of interest.

Expected Fraction of Bin Filled

$(f_n = E[Y_n] \div I)$

| Rejection Probability | Exponential Demand | | Uniform Demand | |
|---|---|---|---|---|
| $\bar{F}(I)$ | $f_1$ | $f_\infty$ | $f_1$ | $f_\infty$ |
| 0.00 | - | - | 0.76 | 0.75 |
| 0.05 | 0.74 | 0.75 | 0.74 | 0.74 |
| 0.10 | 0.69 | 0.70 | 0.72 | 0.72 |
| 0.15 | 0.65 | 0.66 | 0.68 | 0.69 |
| 0.20 | 0.60 | 0.62 | 0.64 | 0.66 |
| 0.25 | 0.56 | 0.58 | 0.60 | 0.62 |

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